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Synchronism in symmetric hyperchaotic systems

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We demonstrate that for symmetric dynamical systems with an invariant subspace in which there is a chaotic attractor, synchronism between the transverse subsystem and its replica can be achieved in wide parameter regimes. The synchronism occurs in situations where the interaction between the invariant subsystem and the transverse subsystem can be either unidirectional or bidirectional, and the full system can possess more than one positive Lyapunov exponent. The idea is illustrated by a numerical example. [S1063-651X(97)51405-X]

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Since the pioneering work of Pecora and Carroll [1], synchronization in chaotic systems has become a field of intense interest [1–4]. The ability for chaotic systems to synchronize with each other provides a possible approach to transmit information via a chaotic carrier and, therefore, it is potentially relevant to applications in communication [2,5]. Naively, synchronous chaos seems quite counterintuitive because chaotic trajectories diverge from each other exponentially. It was shown by Pecora and Carroll [1] that when an appropriately chosen state variable of a chaotic system is used to drive a subsystem (the ''slave''), the subsystem synchronizes with its replica if its Lyapunov exponents are all negative. Given a chaotic system, whether the Pecora-Carroll type of synchronism occurs depends on a proper decomposition of the system into the driving and enslaving subsystems. In this regard, one usually tests various combinations of a subset of state variables to look for a subsystem that possesses only negative Lyapunov exponents.

An important research topic of current interest is how to synchronize chaotic systems with more than one positive Lyapunov exponent (hyperchaotic systems) [4]. This is potentially relevant to applications in secure communication because it is substantially more difficult to extract information from an intercepted hyperchaotic signal. In this paper, we demonstrate that hyperchaotic systems with a simple type of symmetry naturally possess synchronizable subsystems. Simple symmetry often leads to the existence of a (several) low-dimensional invariant subspace(s) (denoted by **S**). Here by ''invariant'' we mean that a trajectory starting in **S** remains in **S** forever. The full phase space can thus be decomposed into the invariant subspace **S** and the subspace **T** that is *transverse* to **S**. Equivalently, the system can be physically decomposed into an invariant subsystem that lives in **S** and a transverse subsystem that lives in **T**. The invariant and transverse subsystems are coupled to each other. We shall argue that when the invariant subsystem is unstable with respect to perturbations in the transverse direction, the transverse subsystem **T** can be *synchronized with its replica* $\mathbf{T}'$ under fairly general conditions, although generally, trajectories in **T** would not synchronize to trajectories in **S**. A practically appealing feature of this class of synchronism is that it can be achieved with as few as one driving signal.

More specifically, the setup of the problem is as follows. Assume there is a chaotic or hyperchaotic attractor in **S**. For randomly chosen initial conditions in **S**, the trajectories live on the chaotic attractor forever because **S** is invariant. For initial conditions off **S**, the resulting trajectories may or may not asymptote to the chaotic attractor in **S**, depending on whether typical trajectories restricted to lie in the chaotic attractor in **S** are transversely stable or transversely unstable, respectively. Now regard the dynamical variables in the transverse subspace **T** as the subsystem to be synchronized. If **S** is transversely stable, synchronism in **T** is trivial because almost all trajectories asymptote to **S**. However, when **S** *is transversely unstable*, synchronism can generally be achieved between the transverse subsystem and its replica when there is a coupling between dynamical variables in **S** and those in **T**. The number of coupled signals from **S** to **T** (the driving signals) can be made as few as one.

There are several features associated with the above described synchronism: (1) the full system, which consists of the invariant subsystem and the transverse subsystem, can possess more than one positive Lyapunov exponent; (2) the interaction between **S** and **T** can be both unidirectional from **S** to **T** and bidirectional; (3) the synchronism occurs in wide regions of positive Lebesgue measure in the parameter space; and (4) the synchronism is robust against small random noise. Due to these features, we expect this type of chaotic synchronism to be observable and constructable in physical systems with symmetry and, consequently, to be useful in practical applications. We stress that our synchronism is *fundamentally different* from synchronization of coupled chaotic oscillators [6]. In that case, there is an invariant subspace corresponding to this synchronous manifold on which all oscillators evolve identically. As such, synchronism among the coupled oscillators occurs when this manifold is *transversely stable*. In our case, synchronism occurs in the transverse subspace and it occurs when the invariant subspace is *transversely unstable*.

We consider a chaotic system $d\mathbf{z}/dt=\mathbf{F}(\mathbf{z})$ with a symmetric invariant subspace **S**, where $\mathbf{z}\in\mathbf{R}^N$ is the state variable and N is the phase-space dimension. Now decompose **z** into two components: $\mathbf{z}\equiv(\mathbf{x},\mathbf{y})$, where $\mathbf{x}\in\mathbf{S}(\mathbf{R}^{N_x})$ represents the invariant subsystem, $\mathbf{y}\in\mathbf{T}(\mathbf{R}^{N_y})$ is the transverse subsystem to be synchronized, and $N_x+N_y=N$. The Pecora-Carroll type of synchronism occurs when the largest Lyapunov exponent of the subsystem **y** is negative [7]. In

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this case, the subsystem $\mathbf{y}$ synchronizes with its replica $\mathbf{y}'$ in the sense that $|\mathbf{y}-\mathbf{y}'|\to 0$ as $t\to\infty$ if both $\mathbf{y}$ and $\mathbf{y}'$ are driven by the same $\mathbf{x}$. For concreteness we consider the following equations for $\mathbf{x}$ and $\mathbf{y}$:

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x},\mathbf{y}),$$
$$\frac{d\mathbf{y}}{dt}=\mathbf{G}(\overline{\mathbf{x}},\mathbf{y}), \tag{1}$$

where $\mathbf{G}(\overline{\mathbf{x}},\mathbf{0})=\mathbf{0}$ so that $\mathbf{y}=\mathbf{0}$ is the invariant subspace $\mathbf{S}$, $\overline{\mathbf{x}}\subset\mathbf{x}$ is a subset of the dynamical variables in $\mathbf{S}$. The function $\mathbf{g}(\mathbf{x},\mathbf{y})$ satisfies $\mathbf{g}(\mathbf{x},\mathbf{0})=\mathbf{0}$ so that the dynamics in the invariant is described by $d\mathbf{x}/dt=\mathbf{f}(\mathbf{x})$ which generates a chaotic attractor with more than one positive Lyapunov exponent. Since in general, $\mathbf{g}(\mathbf{x},\mathbf{y})\neq 0$ off $\mathbf{S}$, the coupling between $\mathbf{x}$ and $\mathbf{y}$ is bidirectional [the case where $\mathbf{g}(\mathbf{x},\mathbf{y})=0$ is called ''unidirectional'' coupling]. The replica of the transverse subsystem to be synchronized is

$$\frac{d\mathbf{y}'}{dt}=\mathbf{G}(\overline{\mathbf{x}},\mathbf{y}'). \tag{2}$$

The largest Lyapunov exponent of the $\mathbf{y}$ subsystem is given by

$$\Lambda_{\rm sub}=\lim_{t\to\infty}\frac{1}{t}\ln\frac{|\delta\mathbf{y}(t)|}{|\delta\mathbf{y}(0)|},$$

where

$$\frac{d\,\delta\mathbf{y}(t)}{dt}=\frac{\partial\mathbf{G}(\overline{\mathbf{x}},\mathbf{y})}{\partial\mathbf{y}}\cdot\delta\mathbf{y}(t). \tag{3}$$

In Eq. (3), the partial derivative $[\partial\mathbf{G}(\overline{\mathbf{x}},\mathbf{y})/\partial\mathbf{y}]$ is the Jacobian matrix evaluated along a trajectory $\{\mathbf{x}(t),\mathbf{y}(t)\}$. Write $\delta\mathbf{y}(t)=\mathbf{y}(t)-\mathbf{y}'(t)$. In the noiseless situation, synchronism in the $\mathbf{y}$ subsystem occurs when $\Lambda_{\rm sub}<0$ so that $\delta\mathbf{y}(t)\to 0$ asymptotically. Since the driving dynamics is invariant in $\mathbf{y}=\mathbf{0}$, there are two cases for synchronization in the $\mathbf{y}$ subsystem, depending on the system parameters. One is the situation where the $\mathbf{x}$ dynamics is transversely stable so that $\mathbf{y}=\mathbf{0}$ is an attractor of the full system Eq. (1). In this case, asymptotically we have $\mathbf{y}\to\mathbf{0}$ for different initial conditions. This, obviously, gives a trivial and uninteresting synchronism in $\mathbf{y}$ [8]. The second case corresponds to the situation where the $\mathbf{x}$ dynamics is *transversely unstable*. In this case, the chaotic attractor in the invariant subspace $\mathbf{y}=\mathbf{0}$ is a repeller in the $\mathbf{y}$ subspace and, hence, the $\mathbf{y}$ dynamics is locally chaotic near $\mathbf{y}=\mathbf{0}$. The $\mathbf{y}$ variables of a typical trajectory would therefore exhibit complicated and nontrivial behavior. But if $\Lambda_{\rm sub}<0$, trajectories starting from different initial conditions will be eventually synchronized. The dynamics of $\mathbf{y}$ near the invariant subspace can be quantitatively characterized by the largest transverse Lyapunov exponent [9] defined as follows:

$$\Lambda_T=\lim_{t\to\infty}\frac{1}{t}\ln\frac{|\delta\mathbf{y}(t)|}{|\delta\mathbf{y}(0)|},$$

where

$$\frac{d\,\delta\mathbf{y}(t)}{dt}=\left.\frac{\partial\mathbf{G}(\overline{\mathbf{x}},\mathbf{y})}{\partial\mathbf{y}}\right|_{\mathbf{y}=\mathbf{0}}\cdot\delta\mathbf{y}(t), \tag{4}$$

where the partial derivative $[\partial\mathbf{G}(\overline{\mathbf{x}},\mathbf{y})/\partial\mathbf{y}]|_{\mathbf{y}=\mathbf{0}}$ is now evaluated at $\mathbf{y}=\mathbf{0}$. The chaotic dynamics $\mathbf{x}$ in the invariant subspace is transversely unstable (stable) if $\Lambda_T>0$ (<0) [10]. Nontrivial synchronism in $\mathbf{y}$ occurs if $\Lambda_T>0$ but $\Lambda_{\rm sub}<0$. The main point of the paper is that there are an infinite number of functions $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{G}$ that can be chosen with relative ease to achieve $\Lambda_T>0$ and $\Lambda_{\rm sub}<0$, even if the full system Eq. (1) is high dimensional and possesses more than one positive Lyapunov exponent. This would provide a way to construct *a priori* synchronous chaotic systems.

A particular class of symmetric systems that can generate synchronous chaos can be constructed as follows. Say we choose $\mathbf{G}(\overline{\mathbf{x}},\mathbf{y})=h(\overline{\mathbf{x}},p)\mathbf{H}(\mathbf{y})$, where $h(\overline{\mathbf{x}},p)$ is a scalar function of the driving variable $\overline{\mathbf{x}}$, p is a parameter that can be varied, and $\mathbf{H}(\mathbf{y})$ satisfies $\mathbf{H}(\mathbf{0})=\mathbf{0}$. An infinitesimal vector evolves according to $d\,\delta\mathbf{y}/dt=h(\overline{\mathbf{x}},p)\mathbf{DH}(\mathbf{y})\cdot\delta\mathbf{y}$, where $\mathbf{DH}(\mathbf{y})$ is the Jacobian matrix $\partial\mathbf{H}/\partial\mathbf{y}$. We can choose the function $\mathbf{H}(\mathbf{y})$ such that $\mathbf{DH}(\mathbf{y})|_{\mathbf{y}=\mathbf{0}}=\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The largest transverse Lyapunov exponent is then given by $\Lambda_T=\lim_{t\to\infty}(1/t)\int_0^\infty h[\overline{\mathbf{x}}(t),p]dt=\int h(\overline{\mathbf{x}},p)\rho(\overline{\mathbf{x}})d\mathbf{x}$, where $\rho(\overline{\mathbf{x}})$ is the invariant density of $\overline{\mathbf{x}}$. One can thus identify parameter regimes with $\Lambda_T>0$ by varying p systematically. This can be done with relative ease, as we will see in a subsequent numerical example. On the other hand, if the Jacobian matrix $\mathbf{DH}(\mathbf{y})$ (without setting $\mathbf{y}=\mathbf{0}$) has negative eigenvalues along the trajectory, it is possible to have $\Lambda_{\rm sub}<0$.

We now give a numerical example. We consider the following six-dimensional flow with small additive noise:

$$\frac{dx_1}{dt}=-x_2-x_3+ay+\epsilon\sigma_{x1}(t),$$
$$\frac{dx_2}{dt}=x_1+0.25x_2+x_4+bz^2+\epsilon\sigma_{x2}(t),$$
$$\frac{dx_3}{dt}=3.0+x_1x_3+\epsilon\sigma_{x3}(t),$$
$$\frac{dx_4}{dt}=-0.5x_3+0.05x_4+\epsilon\sigma_{x4}(t), \tag{5}$$
$$\frac{dy}{dt}=z+\epsilon\sigma_y(t),$$
$$\frac{dz}{dt}=-\alpha z-\gamma y^3+(\beta+f_1\sin x_1+f_2\sin x_2)\sin(2\pi y)+\epsilon\sigma_z(t),$$

where in the absence of noise [corresponding to the noise amplitude $\epsilon=0$ in Eq. (5)], the invariant subspace is four dimensional (x_1,x_2,x_3,x_4) defined by $y=0$ and $z=0$, the transverse subsystem to be synchronized is two dimensional (y,z), a, b, α, γ, β, f_1, and f_2 are parameters, and

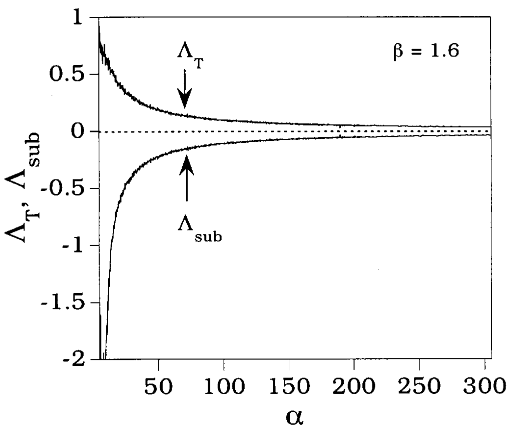

FIG. 1. For the six-dimensional hyperchaotic flow Eq. (5), the transverse Lyapunov exponent Λ_T and the largest Lyapunov exponent $\Lambda_{\rm sub}$ of the transverse subsystem vs α. Other parameters are, β=1.6, $a=1.0$, $b=2.0$, γ=2.0, $f_1=3.5$, and $f_2=5.0$. The positivity of Λ_T and the negativity of $\Lambda_{\rm sub}$ result in nontrivial synchronization of the subsystem with its replica.

$\sigma(t)$'s are random variables with uniform probability density in $[-1,1]$. Note that the case where $a=b=0$ corresponds to unidirectional coupling from the invariant subspace to the transverse subspace, and the coupling terms such as ay and bz^2 are chosen rather arbitrarily. In Eq. (5), the variables (x_1,x_2,x_3,x_4) constitute the hyperchaotic Rössler chaotic system with two positive Lyapunov exponents [11]. The replica of the transverse subsystem to be synchronized is

$$\frac{dy'}{dt}=z'+\epsilon\sigma_{y'}(t),$$

$$\frac{dz'}{dt}=-\alpha z'-\gamma(y')^3+(\beta+f_1\sin x_1+f_2\sin x_2)\sin(2\pi y')+\epsilon\sigma'_z(t). \quad (6)$$

For concreteness we fix $a=1.0$, $b=2.0$, γ=2.0, $f_1=3.5$, $f_2=5.0$, and change α and β to identify synchronizable parameter regimes with $\Lambda_T>0$ and $\Lambda_{\rm sub}<0$ in the noiseless case. Figure 1 shows for β=1.6, Λ_T and $\Lambda_{\rm sub}$ versus α. We see that in the α interval shown, the local dynamics near the invariant subspace is always chaotic because $\Lambda_T>0$, but the global dynamics in the transverse subspace is nonchaotic, rendering synchronizable the transverse subsystem with its replica. Figure 2(a) shows for α=10 and β=1.6, the plots of $z(t)$ versus t and $z'(t)$ versus t, where the trajectories in (y,z) and in (y',z') start from two different random initial conditions. We see that $z(t)$ and $z'(t)$ approach each other rapidly. Figure 2(b) shows, on a semilogarithmic scale, $\Delta(t)\equiv\sqrt{[y(t)-y'(t)]^2+[z(t)-z'(t)]^2}$ versus t for the same parameter setting as in Fig. 2(a), where the noise amplitude is $\epsilon=10^{-12}$. Clearly, trajectories of the transverse subsystem and of its replica approach to each other exponentially to within distances on the order of the noise amplitude. The synchronism is therefore robust in the presence of small noise.

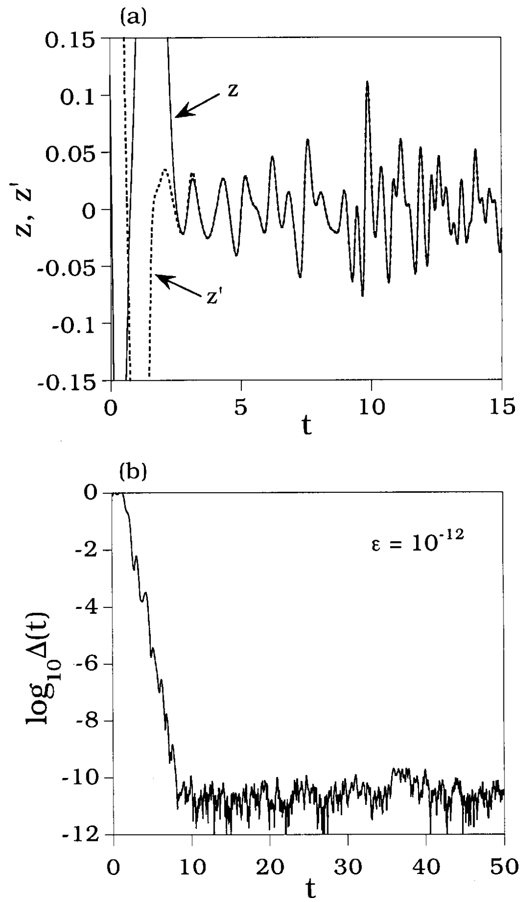

FIG. 2. An illustration of the synchronization for $\alpha=10$ (other parameters are the same as in Fig. 1). Uniform noise of amplitude 10^{-12} is added to Eq. (5). (a) The time series $z(t)$ and $z'(t)$, and (b) $\log_{10}\Delta(t)$ vs t.

We stress four basic aspects associated with the chaotic synchronism in our example Eq. (5): (i) We find numerically that the synchronism can be achieved in wide regions in the two-dimensional parameter space (α,β) and, thus, it is expected to be practically realizable; (ii) We calculate that the full six-dimensional dynamical system at the parameter setting in Figs. 2(a) and 2(b) has the following Lyapunov spectrum (approximately): (0.109, 0.021, 0, -1.891, -7.749, -24.450) and, hence, the synchronism illustrated by Figs. 2(a) and 2(b) occurs for a situation where there are two positive Lyapunov exponents (hyperchaos); (iii) The parameters a and b are not zero so that the interaction between the invariant subsystem and the transverse subsystem is bidirectional. In fact, synchronism is observed for many random choices of a and b, including the unidirectional coupling case where $a=0$ and $b=0$ and; (iv) The synchronism appears to be immune to small external noise.

In summary, we demonstrate numerically and argue theoretically that chaotic synchronism with hyperchaotic driving signals can be realized in high-dimensional dynamical systems when symmetry and invariant subspace are built into the system. In principle, the systems so constructed can have an arbitrary number of positive Lyapunov exponents. The synchronism so designed is robust and occurs in a wide range of parameter values which are easy to identify by considering the transverse stability of the chaotic process in the invariant subspace. These features may be advantageous in practical situations.

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